MODULE - I

DISCRETE PROBABILITY DISTRIBUTION

RANDOM EXPERIMENT

A random experiment is an experiment or a process for which the outcome cannot be predicted with certainity

OUTCOME

An outcome is a possible result of an experiment. Each possible outcome of a particular experiment is unique.

SAMPLE SPACE

Sample space of a random experiment is the set of all possible outcomes

EVENT

An event is a subset of the sample space.

RANDOM VARIABLE

A random variable is a real valued function defined over the samplespace In otherwords a random variable is a mapping from samplespace to real numbers. It is denoted by X

Random variables are of two types

i) Discrete random variable :- A random variable which can take finite number of values or countably infinite number of values

ii) Continuous random variable :- A random variable which can assume any value with in an interval.

PROBABILITY MASS FUNCTION

Let X be a discrete random variable assuming the values $x_1, x_2, \ldots$ from the real line. Let the corresponding probabilities be $f(x_1), f(x_2) \ldots\ldots$ Then $P(X = x_i) = f(x_i)$ is called probability mass function. or probability function of X provided it satisfy the conditions.

i) $f(x_i) \geq 0$ for all i

ii) $\sum f(x_i) = 1$

I. Check whether the following functions can be the probability mass function of a random variable X

1. $f(x) = x-3$, $x = 1, 2, 3, 4, 5$

for $x=1$ $f(1) = -2 < 0$

$\therefore$ the given function is not a probability mass function

2. $f(x) = \frac{x^2}{30}$ $x = 0, 1, 2, 3, 4$

$f(0) = 0$ $f(1) = \frac{1^2}{30} = \frac{1}{30}$ $f(2) = \frac{2^2}{30} = \frac{4}{30}$ $f(3) = \frac{3^2}{30} = \frac{9}{30}$

$f(4) = \frac{4^2}{30} = \frac{16}{30}$

$f(x_i) \geq 0$ $i = 0, 1, 2, 3, 4$

$$\sum_x f(x) = f(0) + f(1) + f(2) + f(3) + f(4)$$

$$= 0 + \frac{1}{30} + \frac{4}{30} + \frac{9}{30} + \frac{16}{30} = \frac{30}{30} = 1$$

$$\sum_x f(x) = 1$$

Two conditions are satisfied, hence the given function is a probability mass function.

U.Q

3. Given that $f(x) = \frac{k}{2^x}$ $x = 0, 1, 2, 3, 4$ is a probability function find k

We have $\sum_{all\,x} f(x) = 1$

$f(0) = \frac{k}{2^0} = k$ $f(1) = \frac{k}{2^1}$ $f(2) = \frac{k}{2^2} = \frac{k}{4}$

$f(3) = \frac{k}{2^3} = \frac{k}{8}$ $f(4) = \frac{k}{2^4} = \frac{k}{16}$

$$\sum_{all\,x} f(x) = 1 \Rightarrow k + \frac{k}{2} + \frac{k}{4} + \frac{k}{8} + \frac{k}{16} = 1$$

$$\Rightarrow k\left[1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\right]=1$$

$$\Rightarrow k \times \frac{31}{16}=1 \quad \Rightarrow k=\frac{16}{31}$$

U.Q
4. A discrete random variable X be a probability function

X	1	2	3	4	5	6	7
$f(x)$	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

i) Find k ii) Evaluate $P(X<3)$, $P(X\geq 6)$, $P(X\leq 6)$

$$\sum_x f(x)=1 \Rightarrow k+2k+2k+3k+k^2+2k^2+7k^2+k=1$$

$$\Rightarrow 10k^2+9k=1 \Rightarrow 10k^2+9k-1=0$$

$$\Rightarrow k=\frac{1}{10} \text{ or } -1$$, here k cannot be $-$ve.

$$\therefore k=\frac{1}{10}$$

X	1	2	3	4	5	6	7
$f(x)$	$\frac{1}{10}$	$1/5$	$1/5$	$3/10$	$1/100$	$1/50$	$17/100$

$P(X=x)=f(x)$.

$$P(X<3)=P(X=1)+P(X=2)$$

$$=\frac{1}{10}+\frac{1}{5}=\frac{3}{10}$$

$$P(X\geq 6)=P(X=6)+P(X=7)$$

$$=\frac{1}{50}+\frac{17}{100}=\frac{19}{100}$$

$$P(X\leq 6)=P(X=1)+P(X=2)+P(X=3)+P(X=4)+P(X=5)+P(X=6)$$

$$=\frac{83}{100}$$

OR

$$P(X\leq 6)=1-P(X>6)=1-P(X=7)$$

$$=1-\frac{17}{100}=\frac{83}{100}$$

5. A random variable X has a probability mass function

$f(x)=k\left(\frac{2}{3}\right)^x \quad x=1,2,3,\ldots$ Find k

$$\sum_x f(x) = 1 \Rightarrow k\cdot\left(\frac{2}{3}\right)^1 + k\cdot\left(\frac{2}{3}\right)^2 + k\cdot\left(\frac{2}{3}\right)^3 + \cdots\cdots = 1$$

$$\Rightarrow k\cdot\frac{2}{3}\left[1 + \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \cdots\cdots\right] = 1$$

$$\Rightarrow k\cdot\frac{2}{3}\left[1 - \frac{2}{3}\right]^{-1} = 1$$

$$\Rightarrow \frac{2k}{3}\left(\frac{1}{3}\right)^{-1} = 1 \qquad (1-x)^{-1} = 1 + x + x^2 + \cdots$$

$$\Rightarrow \frac{2k}{3} \times 3 = 1 \Rightarrow k = \underline{1/2}$$

DISTRIBUTION FUNCTION / CUMULATIVE DISTRIBUTION FUNC:

Let X be a random variable defined on a sample space S then the function $F : R \to R$ defined by $F(x) = P(X \leq x)$ is called the distribution function or cumulative distribution function of the random variable X.

1. Find the distribution function for the following p.m.f

$$f(x) = \frac{1}{4} \quad \text{if } x = 0, 2$$
$$= \frac{1}{2} \quad \text{if } x = 1$$
$$= 0 \quad \text{otherwise.}$$

Probability function

X	0	1	2
$f(x)$	1/4	1/2	1/4

X takes values 0, 1, 2

$$F(x) = P(X \leq x)$$

$x = 0$, $F(0) = P(X \leq 0) = P(X=0) = \frac{1}{4}$

$x = 1$, $F(1) = P(X \leq 1) = P(X=0) + P(X=1) = \frac{1}{4} + \frac{1}{2} = \frac{3}{4}$

$x = 2$, $F(2) = P(X \leq 2) = P(X=0) + P(X=1) + P(X=2) = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$

Distribution function

X	0	1	2
$F(x)$	1/4	3/4	1

2. If X is a random variable taking the values 1, 2, 3, 4 & $P(X=x) = \frac{x}{10}$. Find the distribution function

Probability function

X	1	2	3	4
$f(x)$	$1/10$	$2/10$	$3/10$	$4/10$

X takes values $1, 2, 3, 4$

$$F(x) = P(X \le x)$$

$$x = 1 \Rightarrow F(1) = P(X \le 1) = P(X=1) = 1/10$$

$$x = 2 \Rightarrow F(2) = P(X \le 2) = P(X=1) + P(X=2) = 3/10$$

$$x = 3 \Rightarrow F(3) = P(X \le 3) = P(X=1) + P(X=2) + P(X=3) = 6/10$$

$$x = 4 \Rightarrow F(4) = P(X \le 4) = P(X=1) + P(X=2) + P(X=3) + P(X=4) = 1$$

Distribution function

X	1	2	3	4
$F(x)$	$1/10$	$3/10$	$6/10$	1

VQ 3. If a random variable X takes the values $1, 2, 3, 4$ such that $2P(X=1) = 3P(X=2) = P(X=3) = 5P(X=4)$. Find the probability distribution. Also find the distribution function.

X takes values $1, 2, 3, 4$

Let $P(X=3) = k \Rightarrow P(X=1) = \frac{k}{2}$, $P(X=4) = \frac{k}{5}$, $P(X=2) = \frac{k}{3}$

$$\because \sum_x f(x) = 1 \Rightarrow \frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$$

$$\Rightarrow k\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{5}\right) = 1$$

$$\Rightarrow k \cdot \frac{61}{30} = 1 \Rightarrow k = \frac{30}{61}$$

Probability function

X	1	2	3	4
$f(x)$	$15/61$	$\frac{10}{61}$	$\frac{30}{61}$	$\frac{6}{61}$

$$F(x) = P(X \le x)$$

$$F(1) = P(X \le 1) = P(X=1) = \frac{15}{61}$$

$$F(2) = P(X \le 2) = P(X=1) + P(X=2) = \frac{25}{61}$$

$$F(3) = P(X \le 3) = \frac{55}{61}$$

$$F(4) = P(X \le 4) = 1$$

Distribution function.

X	1	2	3	4
$F(x)$	$\frac{15}{61}$	$\frac{25}{61}$	$\frac{55}{61}$	1

4. A random variable X takes the values $-1, 1, 3$ with equal probabilities & 5 with probability ½

1) Find the probability distribution of X

2) Find $P(|X-3|>1)$

X takes values $-1, 1, 3, 5$

1) Given $P(X=-1) = P(X=1) = P(X=3) = k$

$$P(X=5) = \frac{1}{2}$$

$$\sum_x f(x) = 1 \Rightarrow k+k+k+\frac{1}{2} = 1 \Rightarrow 3k = \frac{1}{2} \Rightarrow k = \frac{1}{6}$$

X	-1	1	3	5
$f(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$

$|X|<1 \Rightarrow -1<X<1$

2. $$P(|X-3|>1) = 1 - P(|X-3| \leq 1)$$

$$= 1 - P\{-1 \leq X-3 \leq 1\}$$

$$= 1 - P\{2 \leq X \leq 4\}$$

$$= 1 - P(X=3)$$

$$= 1 - \frac{1}{6} = \frac{5}{6}$$

5. Obtain the distribution function of the total no: of heads occuring in 3 tosses of an unbiased coin

X denotes no: of heads

X takes values $0, 1, 2, 3$

$$P(X=0) = P(TTT) = \frac{1}{8}$$

$$P(X=1) = P(HTT) + P(THT) + P(TTH)$$

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

$$P(X=2) = P(HHT) + P(HTH) + P(THH) = \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$$

$$P(X=3) = P(HHH) = \frac{1}{8}$$

Probability distribution

X	0	1	2	3
$f(x)$	$1/8$	$3/8$	$3/8$	$1/8$

$F(x) = P(X \leq x)$

$F(0) = P(X \leq 0) = P(X=0) = \frac{1}{8}$ $F(2) = P(X \leq 2) = \frac{7}{8}$

$F(1) = P(X \leq 1) = P(X=0) + P(X=1) = \frac{4}{8}$ $F(3) = P(X \leq 3) = 1$

Distribution function

X	0	1	2	3
$F(x)$	$1/8$	$\frac{4}{8}$	$\frac{7}{8}$	1

6. If the probability distribution of a discrete random variable X defined as $P(X=x) = \begin{cases} \frac{x}{15} & x = 1,2,3,4,5 \\ 0 & \text{otherwise} \end{cases}$

Find i) $P(1 \text{ or } 2)$ ii $P\left(\frac{1}{2} < X < \frac{5}{2} \Big/ X > 1\right)$

X takes values $1, 2, 3, 4, 5$

$P(X=1) = \frac{1}{15}$ $P(X=2) = \frac{2}{15}$ $P(X=3) = \frac{3}{15}$ $P(X=4) = \frac{4}{15}$

$P(X=5) = \frac{5}{15}$

1) $P(1 \text{ or } 2) = P(X=1) + P(X=2) = \frac{1}{15} + \frac{2}{15} = \frac{3}{15} = \frac{1}{5}$

ii) $P\left(\frac{1}{2} < X < \frac{5}{2} \Big/ X > 1\right) = \frac{P\left\{\left(\frac{1}{2} < X < \frac{5}{2}\right) \cap (X > 1)\right\}}{P(X>1)}$

$P(A/B) = \frac{P(A \cap B)}{P(B)}$

$= \frac{P\left(1 < X < \frac{5}{2}\right)}{P(X>1)} = \frac{P(X=2)}{1 - P(X=1)}$

$= \frac{2/15}{1 - \frac{1}{15}} = \frac{1}{7}$

7. Consider a lot of 10 items containing 3 defectives from which a sample of 4 items is drawn at random. Let the random variable X denote the no: of defective item in the sample. Find the distribution function

X denotes the no: of defective items

X takes values 0, 1, 2, 3

$P(X=0) = \frac{7C_4 \; 3C_0}{10C_4} = \frac{1}{6}$ $\quad P(X=1) = \frac{7C_3 \; 3C_1}{10C_4} = \frac{1}{2}$

$P(X=2) = \frac{7C_2 \; 3C_2}{10C_4} = \frac{3}{10}$ $\quad P(X=3) = \frac{7C_1 \; 3C_3}{10C_4} = \frac{1}{30}$

X	0	1	2	3
f(x)	1/6	1/2	3/10	1/30

$F(x) = P(X \le x)$

$F(0) = P(X \le 0) = P(X=0) = \frac{1}{6}$ $\quad F(1) = P(X \le 1) = P(X=0) + P(X=1) = \frac{2}{3}$

$F(2) = \frac{29}{30}$ $\quad F(3) = 1$

X	0	1	2	3
F(x)	1/6	2/3	29/30	1

8. Two bad eggs are mixed accidently with 10 good ones. Find the probability distribution of no: of bad eggs in 3 drawn at random without replacement from this lot.

$X \rightarrow$ No: of bad eggs

X takes values 0, 1, 2,

$P(X=0) = \frac{10C_3 \; 2C_0}{12C_3} = \frac{6}{11}$ $\quad P(X=1) = \frac{10C_2 \; 2C_1}{12C_3} = \frac{9}{22}$

$P(X=2) = \frac{10C_1 \; 2C_2}{12C_3} = \frac{1}{22}$

Probability distribution

X	0	1	2	
$f(x)$	$6/11$	$9/22$	$1/22$	

9. The probability distribution function of a random variable X is

X	0	1	2	3	4	5	6	
$f(x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$	

1. i) Find $P(X<4)$ ii) $P(X\geq 5)$
2. What will be the minimum value of k so that $P(X\leq 2) > 0.3$

$$\sum_x f(x) = 1$$

$$k+3k+5k+7k+9k+11k+13k = 1 \Rightarrow 49k = 1$$
$$\Rightarrow k = 1/49.$$

i) $P(X<4) = P(X=0)+P(X=1)+P(X=2)+P(X=3)$

$$= \frac{1}{49} + \frac{3}{49} + \frac{5}{49} + \frac{7}{49} = \frac{16}{49}$$

ii) $P(X\geq 5) = P(X=5)+P(X=6) = \frac{11}{49} + \frac{13}{49} = \frac{24}{49}$

2. $P(X\leq 2) > 0.3$

$$k+3k+5k > 0.3 \Rightarrow 9k > 0.3 \Rightarrow k > \frac{0.3}{9} \Rightarrow k > \frac{1}{30}$$

MEAN AND VARIANCE

If X is a discrete random variable that takes on values $x_1, x_2 \ldots x_n$ with probabilities $f(x_1), f(x_2), \ldots f(x_n)$ then its mean or mathematical expectation $E[X]$ is defined as Mean, $\mu = E[X] = \sum_{\text{all } x} x f(x)$

Variance, $V(X) = \sigma^2 = E[X^2] - E[X]^2$ $\qquad E[X^2] = \sum x^2 f(x)$

Properties

1. $E[aX+b] = aE[X]+b$
2. $V[aX+b] = a^2 V[X]$

1 Suppose that the probabilities are 0.4, 0.3, 0.2, 0.1 that there will be 0, 1, 2, or 3 power failure in certain city during the month of July. Find the mean & variance of this probability distributions.

X	0	1	2	3
f(x)	0.4	0.3	0.2	0.1

Mean, $\mu = E[X] = \sum x f(x) = 0 \times 0.4 + 1 \times 0.3 + 2 \times 0.2 + 3 \times 0.1 = 1$

$V(X) = E[X^2] - E[X]^2$ $\quad E[X^2] = \sum x^2 f(x)$

$= 2 - 1^2 = \underline{\underline{1}}$ $\quad = 0^2 \times 0.4 + 1^2 \times 0.3 + 2^2 \times 0.2 + 3^2 \times 0.1$

$= 2.$

2. Let X be a random variable of the following probability distribution

X	-3	6	9
f(x)	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

1. Find 1) E(X)
2. $E(2X+1)^2$

$E[X] = \sum x f(x) = -3 \times \frac{1}{6} + 6 \times \frac{1}{2} + 9 \times \frac{1}{3} = \frac{11}{2}$

$E[2X+1]^2 = E[4X^2 + 4X + 1] = 4E[X^2] + 4E[X] + 1$

$E[X^2] = \sum x^2 f(x) = (-3)^2 \times \frac{1}{6} + 6^2 \times \frac{1}{2} + 9^2 \times \frac{1}{3} = \frac{93}{2}$

$E[2X+1]^2 = 4 \times \frac{93}{2} + 4 \times \frac{11}{2} + 1 = 209$

3. Let X be a random variable of the following probability distribution

X	4	5	6	7	8	9
f(x)	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

Find the mean & variance of 2X−1 & also its std. deviation

$E[X] = \sum x f(x) = \frac{41}{6}$ $\quad E[X^2] = \sum x^2 f(x) = \frac{293}{6}$

$E(2X-1) = 2E[X] - 1 = 2 \times \frac{41}{6} - 1 = \frac{38}{3}$

$V(2X-1) = 2^2 V(X) = 4V(X)$

$V(X) = E(X^2) - E(X)^2 = \frac{293}{6} - \frac{1681}{36}$

$V(2X-1) = \frac{77}{9} = 8.56$ $\quad$ Std. deviation $= +\sqrt{V(X)} = 2.925$

4. 20 fair dice are thrown. Find the expectation of the sum of the numbers thrown.

$X \rightarrow$ Sum of the no:s thrown

$$E[X] = E[X_1 + X_2 + \cdots + X_{20}] = E[X_1] + E[X_2] + \cdots + E[X_{20}]$$
$$= 20\,E[X_1]$$

$E[X_1] = \sum x\, f(x)$

X	1	2	3	4	5	6
f(x)	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

$E[X_1] = \sum x\, f(x) = \frac{7}{2}$

$E[X] = 20 \times \frac{7}{2} = \underline{\underline{70}}$

5. A player is to toss 3 coins. He wins rupees 10 if 3 heads appear, Rs 5 if 2 heads appear Rs 1 if 1 head appears. He will lose Rs 12 if no heads appear. What does the expected amount.

$X \rightarrow$ amount $\quad$ X takes values 10, 1, 5, −12

X	10	5	1	−12	
f(x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$.	

$P(\text{3heads}) = P(HHH) = \frac{1}{8}$

$P(\text{2heads}) = P(HTH) + P(THH) + P(HHT)$
$= \frac{3}{8}$

$P(\text{1 head}) = P(THT) + P(TTH) + P(HTT)$
$= \frac{3}{8}$.

$P(\text{No head}) = P(TTT) = \frac{1}{8}$.

$E[X] = \sum x\, f(x) = 10 \times \frac{1}{8} + 5 \times \frac{3}{8} + 1 \times \frac{3}{8} - 12 \times \frac{1}{8} = \underline{\underline{2}}$

6. Find the mean of the probability distribution of the no: of heads obtained in 2 toses of an unbiased coin.

$X \rightarrow$ No: of heads $\quad$ X takes values 0, 1, 2

$P(X=0) = P(TT) = \frac{1}{4}$ $\quad$ $P(X=1) = P(HT) + P(TH) = \frac{2}{4} = \frac{1}{2}$

$P(X=2) = P(HH) = \frac{1}{4}$

X	0	1	2
f(x)	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

$E[X] = \sum x\, f(x) = 0 \times \frac{1}{4} + 1 \times \frac{1}{2} + 2 \times \frac{1}{4}$

$= \underline{\underline{1}}$.

BINOMIAL DISTRIBUTION

A discrete random variable X is said to follow the binomial distribution then the p.m.f of X is given by

$$f(x) = P(X=x) = {}^{n}C_{x}\, p^{x} q^{n-x} \quad \text{for } x = 0,1,2\ldots n \quad \text{also } p+q=1, \; 0 \le p \le 1$$

where n & p are the parameters & n is the number of trials, p is the probability of success. It is denoted as $b(x;n,p)$ or $B(x;n,p)$

MEAN AND VARIANCE OF BINOMIAL DISTRIBUTION

Mean $= E[X] = np$

Variance, $= npq$.

⇒ **Mean**

$$E[X] = \sum_{x=0}^{n} x\, f(x) = \sum_{x=0}^{n} x \cdot {}^{n}C_{x}\, p^{x} q^{n-x}.$$

$$= \sum_{x=0}^{n} x \cdot \frac{n!}{x!\,(n-x)!}\, p^{x} q^{n-x}.$$

$$= \sum_{x=0}^{n} x \cdot \frac{n!}{x(x-1)!\,(n-x)!}\, p^{x} q^{n-x}$$

$$= \sum_{x=0}^{n} \frac{n!}{(x-1)!\,(n-x)!}\, p^{x} q^{n-x}.$$

$$= \sum_{x=0}^{n} \frac{n(n-1)!}{(x-1)!\,(n-x)!}\, p \cdot p^{x-1} q^{n-x}.$$

$$= np \sum_{x=0}^{n} \frac{(n-1)!}{(x-1)!\,(n-x)!}\, p^{x-1} q^{n-x}$$

$$= np \sum_{x=0}^{n} \frac{(n-1)!}{(x-1)!\,[(n-1)-(x-1)]!}\, p^{x-1} q^{(n-1)-(x-1)}$$

$$= np \sum_{x=0}^{n} {}^{n-1}C_{x-1}\, p^{x-1} q^{(n-1)-(x-1)}$$

$$= np\,[p+q]^{n-1} = \cancel{np \times 1}$$

$$= \underline{np}.$$

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VARIANCE

$$V(X) = E[X^2] - E[X]^2 \qquad E[X] = np.$$

$$E[X^2] = E[X(X-1) + X]$$

$$= E[X(X-1)] + E[X]$$

$$= \sum_{x=0}^{n} x(x-1)f(x) + np.$$

$$= \sum_{x=0}^{n} x(x-1)\ {}^{n}C_x\, p^x q^{n-x} + np$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n!}{x!(n-x)!} p^x q^{n-x} + np.$$

$$= \sum_{x=0}^{n} \frac{x(x-1)\ n!}{x(x-1)(x-2)!(n-x)!} p^x q^{n-x} + np$$

$$= \sum_{x=2}^{n} \frac{n(n-1)(n-2)!}{(x-2)!(n-x)!} p^2 \cdot p^{x-2} q^{n-x} + np$$

$$= \sum_{x=2}^{n} n(n-1)p^2 \cdot \frac{(n-2)!}{(x-2)!(n-x)!} p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!(n-x)!} p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} {}^{n-2}C_{x-2}\, p^{x-2} q^{n-x} + np.$$

$$= n(n-1)p^2 [p+q]^{n-2} + np.$$

$$= n(n-1)p^2 + np.$$

$$V(X) = E[X^2] - E[X]^2$$

$$= n(n-1)p^2 + np - (np)^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np - np^2$$

$$= np(1-p)$$

$$= \underline{\underline{npq}}$$

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1. Determine the binomial distribution for which the mean is 4 & variance is 3

Mean = 4 $\Rightarrow np = 4$ — ①

Variance = 3 $\Rightarrow npq = 3$ — ②

From ① & ② $4q = 3 \Rightarrow q = \frac{3}{4} \Rightarrow p = 1 - \frac{3}{4} = \frac{1}{4}$

$np = 4 \Rightarrow n \cdot \frac{1}{4} = 4 \Rightarrow n = 16.$

$\therefore f(x) = 16C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{16-x} \quad x = 0, 1, 2 \ldots 16$

2. An unbiased coin is tossed four times. Find the probability of getting i) Exactly one head ii) atmost 3 heads iii) Atleast 2 heads

X denotes no: of heads

X follows binomial distribution with $n = 4 \quad p = \frac{1}{2}$.

$f(x) = P(X = x) = 4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x} \quad x = 0, 1, 2, 3, 4$

i) $P(\text{Exactly one head}) = P(X = 1) = 4C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^3 = \frac{1}{4}$

ii) $P(\text{atmost 3 heads}) = P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

OR

$= 1 - P(X > 3) = 1 - P(X = 4)$

$= 1 - \left[4C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^0\right] = \frac{15}{16}$

iii) $P(\text{Atleast 2 heads}) = P(X \geq 2)$

$= 1 - P(X < 2) = 1 - \{P(X=0) + P(X=1)\}$

$= 1 - \frac{5}{16} = \frac{11}{16}.$

U.Q

3. During one stage in the manufacture of IC chips a coating must be applied. If 70% of chips receive a thick enough coating. Find the probabilities that among 15 chips

i) atleast 12 will have thick enough coating

ii) atmost 6 will have thick enough coating

iii) exactly 10 will have thick enough coating

$X \rightarrow$ No: of chips having thick enough coating

$n = 15 \quad p = 0.7 \rightarrow (70\%)$

$X \sim B(x; 15, 0.7)$

$f(x) = P(X=x) = {}^{15}C_x (0.7)^x (0.3)^{15-x} \quad x = 0, 1, \ldots 15$

i) $P(\text{atleast } 12) = P(X \geq 12) = P(X=12) + P(X=13) + P(X=14) + P(X=15)$

$= 0.294$

ii) $P(\text{atmost } 6) = P(X \leq 6) = 0.0154$

iii) $P(\text{exactly } 10) = P(X=10) = 0.206.$

4. U.Q The probability that a component is acceptable is 0.93. 10 components are picked at random. what is the probability that i) atleast 9 are acceptable. ii) atmost 3 are acceptable.

$n = 10 \quad p = 0.93 \quad X$ denotes no: of components acceptable.

X follows binomial distribution with $n = 10$ & $p = 0.93$

$f(x) = P(X=x) = {}^{10}C_x (0.93)^x (0.07)^{10-x}, \quad x = 0, 1, 2 \ldots 10.$

i) $P(\text{atleast 9 are acceptable}) = P(X \geq 9) = P(X=9) + P(X=10) = 0.84$

ii) $P(\text{atmost 3 are acceptable}) = P(X \leq 3) = 8.177 \times 10^{-7}$

5. U.Q An insurance company agent accepts policies of 5 men all of identical age and good health. Probability that a man of this age will be alive in 30 years is $\frac{2}{3}$. Find the probability that in 30 years i) all 5 men ii) atleast one men will be alive.

$X \rightarrow$ No: of men that will be alive in 30 years. $X \sim B(x; 5, \frac{2}{3})$

$n = 5 \quad p = \frac{2}{3} \quad f(x) = P(X=x) = {}^{5}C_x \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{5-x}.$

$x = 0, 1, 2 \ldots 5$

i) $P(\text{all 5 men}) = P(X=5) = 0.1316$

ii) $P(\text{atleast 1 men}) = P(X \geq 1) = 1 - P(X < 1)$

$= 1 - [P(X=0)] = 0.996.$

6. U.Q If 6 of the 18 new buildings in a city violate the building code what is the probability that a building inspector who randomly select 4 of the new buildings will catch i) none of the new buildings that violate the building code (ii) One of the new building that violate the building code (iii) atleast two of the new buildings violate the building code.

$n = 4 \quad p = \frac{6}{18} = \frac{1}{3}$

X denotes no: of new buildings that violate the building code

$$f(x) = P(X=x) = 4C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{4-x}, \quad x = 0,1,2,3,4$$

i) P(None of the new building) = $P(X=0) = 0.197$

ii) P(One of the ") = $P(X=1) = 0.395$

iii) P(atleast two) = $P(X \geq 2) = 1 - P(X<2) = 0.408$

7. A box contains 100 cell phones, 20 of which are defective. 10 cell phones are selected for inspection. Find the probability that i) Atleast one is defective ii) all the 10 are defective iii) None of them is defective.

$X \to$ No: of defectives $\quad X \sim B(x; 10, \frac{1}{5})$

$n = 10 \quad p = \frac{20}{100} = \frac{1}{5} \quad f(x) = P(X=x) = {}^{10}C_x \left(\frac{1}{5}\right)^x \left(\frac{4}{5}\right)^{10-x}$

$x = 0,1,2,\dots 10$

i) P(atleast one) = $P(X \geq 1) = 1 - P(X<1)$

$= 1 - P(X=0) = 0.8926$

ii) P(all the 10) = $P(X=10) = 1.024 \times 10^{-7}$

iii) P(None of them) = $P(X=0) = 0.\cancel{13420}\ 0.1073$

8. Out of 2000 families with 4 children each, how many would you expect to have a) atleast one boy b) 1 or 2 girls.

$n = 4 \quad p = \frac{1}{2} \quad X \to$ No: of boys

$$P(X=x) = 4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x} \quad x = 0,1,2,3,4$$

a) P(atleast one boy) = $P(X \geq 1) = 1 - P(X<1) = 1 - P(X=0)$

$= \frac{15}{16}$

No: of families expect to have atleast one boy $= 2000 \times \frac{15}{16}$

$= 1875$

b) P(1 or 2 girls) = P(3 or 2 boys) = $P(X=3) + P(X=2)$

$= \frac{5}{8}$

No: of families $= \frac{5}{8} \times 2000 = 1250$

9. In 800 families with 5 children each, how many families would you expect to have a) 3 boys & 2 girls b) 2 boys & 3 girls c) No girls d) atmost 2 girls.

$N = 800$

$n = 5 \quad p = \frac{1}{2} \quad X \to$ No: of girls.

$$f(x) = P(X = x) = {}^5C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{5-x} \quad x = 0, 1, \ldots 5$$

a) $P(3\text{boys} \,\&\, 2\text{ girls}) = P(2\text{girls and } 2\text{girls}) = P(2\text{girls})$
$= P(X = 2) = \frac{5}{16}$

b) $P(2\text{boys} + 3\text{girls}) = P(3\text{girls and } 3\text{girls}) = P(3\text{girls}) = P(X=3) = \frac{5}{16}$

c) $P(\text{No girls}) = P(X=0) = \frac{1}{32}$

d) $P(\text{atmost } 2 \text{ girls}) = P(X \leq 2) = P(X=0) + P(X=1) + P(X=2) = \frac{1}{2}$

a) No: of families $= 250$

b) ,, $= 250$

c) ,, $= 25$

d) ,, $= 400$

10. U.Q A complex electronic system is built with a certain number of backup components in its subsystems. One subsystem has four identical components each with a probability of 0.2 of failing in less than 1000 hours. The subsystem will operate if any two of the four components are operating. Assume that the components operate independently. Find the probability that
i) exactly two of the four components last longer than 1000 hours
ii) the subsystem operates longer than 1000 hours.

$X \to$ No: of components that last longer than 1000 hrs

$n = 4 \quad p = 0.8 \to [1 - 0.2] \quad P(X=x) = {}^4C_x (0.8)^x (0.2)^{4-x} \quad x = 0, 1, 2, 3, 4$

i) $P(\text{Exactly two}) = P(X=2) = 0.1536$

ii) $P(X \geq 2) = 1 - P(X < 2) = 1 - [P(X=0) + P(X=1)] = 0.9728$

11. U.Q 6 dies are thrown 729 times. How many times would you expect atleast 3 dies to show 1 or 2?

$P(1 \text{ or } 2) = p = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

$N = 729 \quad n = 6 \quad p = \frac{1}{3}$

$X \to$ No: of dies that show 1 or 2 $\quad P(X=x) = {}^6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}$

$x = 0, 1, \ldots 6$

$P(\text{atleast } 3) = P(X \geq 3) = 1 - P(X<3) = 1 - [P(X=0)+P(X=1)+P(X=2)]$

$= \frac{233}{729}$

No: of times $= 729 \times \frac{233}{729} = \underline{\underline{233}}$

12. The probability of a man hitting a target is $\frac{1}{3}$. How many times must he try so that the probability of hitting the target atleast one is more than 90%.

$p = \frac{1}{3}$ $\quad P(X \geq 1) > \frac{90}{100} \Rightarrow 1 - P(X<1) > 0.9$

$\Rightarrow 1 - P(X=0) > 0.9$

$\Rightarrow P(X=0) < 0.1$

$\Rightarrow nC_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{n-0} < 0.1$

$\Rightarrow \left(\frac{2}{3}\right)^n < 0.1 \Rightarrow n \log \frac{2}{3} < \log 0.1$

$\Rightarrow n > 5.6$

So he must try atleast 6 times.

POISSON DISTRIBUTION

A random variable X is said to have a poisson distribution with parameter λ if its probability mass function is given by

$$f(x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad \lambda > 0, \quad x = 0, 1, 2 \ldots\ldots$$

MEAN + VARIANCE OF POISSON DISTRIBUTION

MEAN

$$\text{Mean} = E[X] = \sum_{x=0}^{\infty} x f(x)$$

$$= \sum_{x=0} x \cdot \frac{e^{-\lambda} \cdot x^x}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{x\, e^{-\lambda} \cdot \lambda^x}{x(x-1)!} = \sum_{x=\phi}^{\infty} \frac{e^{-\lambda} \cdot \lambda^x}{(x-1)!}$$

$$= e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!} = e^{-\lambda} \left[\frac{\lambda^1}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots\ldots\right]$$

$$= e^{-\lambda} \cdot \lambda \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \ldots\right] = e^{-\lambda} \cdot \lambda \cdot e^{\lambda} = \underline{\underline{\lambda}}$$

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VARIANCE $= E[X^2] - E[X]^2$

$$E[X^2] = E[X(X-1)+X] = E[X(X-1)] + E[X]$$

$$= \sum_{x=0}^{\infty} x(x-1) f(x) + \lambda.$$

$$= \sum_{x=0}^{\infty} x(x-1) \cdot \frac{e^{-\lambda} \cdot \lambda^x}{x!} + \lambda$$

$$= \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda} \cdot \lambda^x}{x(x-1)(x-2)!} + \lambda$$

$$= \sum_{x=2}^{\infty} \frac{e^{-\lambda} \cdot \lambda^x}{(x-2)!} + \lambda.$$

$$= e^{-\lambda} \sum_{x=2}^{\infty} \frac{\lambda^x}{(x-2)!} + \lambda$$

$$= e^{-\lambda} \left[\frac{\lambda^2}{1} + \frac{\lambda^3}{1!} + \frac{\lambda^4}{2!} + \cdots \right] + \lambda$$

$$= e^{-\lambda} \cdot \lambda^2 \left[1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \cdots \right] + \lambda.$$

$$= \lambda^2 \cdot e^{-\lambda} \cdot e^{\lambda} + \lambda = \lambda^2 + \lambda.$$

$$V(X) = E[X^2] - E[X]^2 = \lambda^2 + \lambda - \lambda^2 = \underline{\underline{\lambda}}$$

POISSON DISTRIBUTION AS LIMITING FORM OF BINOMIAL DISTRIBUTION

when $n \to \infty$, $p \to 0$ such that $np = \lambda$ remains a constant the limiting form of binomial distribution is poisson distribution

$\Rightarrow$ $n \to \infty$ $p \to 0$ $np = \lambda$ $p = \frac{\lambda}{n}.$

$$f(x) = {}^nC_x \; p^x q^{n-x}.$$

$$= \frac{n!}{x!(n-x)!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x}.$$

$$= \frac{n(n-1)(n-2)\cdots(n-x+1)(n-x)\cdots 3\times2\times1}{x! \quad (n-x)!} \cdot \frac{\lambda^x}{n^x} \left(1-\frac{\lambda}{n}\right)^{n-x}.$$

$$= \frac{n(n-1)(n-2)\cdots(n-(x-1))(n-x)!}{x! \; (n-x)!} \frac{\lambda^x}{n^x} \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}.$$

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$$= \frac{n}{n} \frac{(n-1)}{n} \frac{(n-2)}{n} \cdots \frac{n-(x-1)}{n} \cdot \frac{\lambda^x}{x!} \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

$$= 1 \times \left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \cdots \left(1-\frac{x-1}{n}\right) \cdot \frac{\lambda^x}{x!} \left(1-\frac{\lambda}{n}\right)^n \left(1-\frac{\lambda}{n}\right)^{-x}$$

when $n \to \infty$ $\quad \frac{1}{n}, \frac{2}{n} \cdots \frac{x-1}{n} \to 0$

Also $\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$ & $\lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^{-x} = 1$

$$f(x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

1. The average no: of phone calls per minute coming into a switchboard between 2 & 4 pm is 2.5. Determine the probability that during one particular minute there will be i) zero ii) 4 or less iii) more than 6 telephone calls.

$\lambda = 2.5$ $\quad X \to$ No: of telephone calls

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad x = 0, 1, 2, 3 \ldots$$

i) $P(X=0) = 0.0821$

ii) $P(X \le 4) = P(X=0)+P(X=1)+P(X=2)+P(X=3)+P(X=4) = 0.8912$

iii) $P(X>6) = 1-P(X \le 6) = 0.0142$

2. U.Q A car hire firm has 2 cars which hires out day by day. The number of demands for a car on each day is distributed as a poisson distribution with mean 1.5. Calculate the proportion of days on which i) neither car is used ii) some demand is refused

X denotes the demand for cars

$\lambda = 1.5$ $\quad f(x) = \frac{e^{-1.5}(1.5)^x}{x!} \quad x = 0, 1, 2, \ldots$

i) $P(\text{neither car}) = P(X=0) = \underline{0.2231}$

ii) $P(\text{Some demand refused}) = P(X>2) = 1-P(X\le 2) = \underline{0.1912}$

U.Q

3. It is known that 5% of the books bound at a bindery have defective bindings. Find the probability that atmost 2 of 100 books bound by this bindery will have defective binding. using poisson distribution

$X \rightarrow$ No: of books having defective bindings

$n = 100 \quad p = 0.05 \quad np = \lambda \quad \lambda = 100 \times 0.05 = \underline{\underline{5}}$

$$P(X=x) = f(x) = \frac{e^{-5}\, 5^x}{x!} \quad x = 0,1,2,\ldots$$

$P(\text{atmost } 2) = P(X \leq 2) = 0.1246$

4. In a factory turning out razor blade, there is a small chance of 1/500 for any blade to be defective. The blades are supplied in a packet of 10. Use Poisson distribution to calculate the approximate number of packets containing blades with no defective, one defective, two defectives blades in a consignment of 10,000 packets.

X denotes number of defective blades

$n = 10 \quad p = \frac{1}{500}$ $\quad X$ follows poisson distribution with $\lambda = np$

$\lambda = np = 0.02$ $\quad P(X=x) = \frac{e^{-0.02}(0.02)^x}{x!} \quad x = 0,1,2\ldots$

i) $P(X=0) = 0.9802$

Number of packets containing no defectives $= 9802$

ii) $P(X=1) = 0.196$

No: of packets containing one defectives $= 196$

iii) $P(X=2) = 0.000196$

No: of packets containing two defective $= 2$

U.Q
5. In a given city 6% of all drivers get atleast one parking ticket per year. Use poisson approximation to the binomial distribution to determine the probabilities that among 80 drivers

i) 4 will get atleast one parking ticket per year

ii) Atleast 3 will get atleast one parking ticket per year.

iii) Anywhere from 3-6, inclusive will get atleast one parking ticket per year.

$X \rightarrow$ No of drivers that will get atleast one parking ticket per year.

$n = 80 \quad p = 0.06 \quad np = \lambda = 80 \times 0.06 = 4.8$

i) $P(X=4) = 0.1820$

ii) $P(\text{atleast } 3) = P(X \geq 3) = 1 - P(X<3) = 0.8076$

iii) $P(3 \leq X \leq 6) = 0.875$

6. The probability that a man aged 50 years will die within a year is 0.01125. What is the probability that of 12 such that atleast 11 will reach their 51^{st} birthday?

$$p = 0.01125 \qquad n = 12 \qquad np = \lambda = 0.01125 \times 12 = 0.135$$

P(atleast 11 will reach their 51^{st} birthday) = P(atmost 1 die)

$$= P(X \leq 1) = P(X=0) + P(X=1)$$

$$f(x) = P(X=x) = \frac{e^{-0.135} \cdot (0.135)^x}{x!} \qquad x = 0, 1, 2 \ldots$$

$$P(X \leq 1) = 0.9916$$

7. Prove that for Poisson distribution $\frac{f(x+1, \lambda)}{f(x, \lambda)} = \frac{\lambda}{x+1}$ where

$$f(x, \lambda) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

$$f(x+1, \lambda) = \frac{e^{-\lambda} \cdot \lambda^{x+1}}{(x+1)!}$$

$$\frac{f(x+1, \lambda)}{f(x, \lambda)} = \frac{e^{-\lambda} \cdot \lambda^{x+1}}{(x+1)!} \cdot \frac{x!}{e^{-\lambda} \cdot \lambda^x}$$

$$= \frac{\lambda^x \cdot \lambda \cdot x!}{(x+1)\, x!\, \lambda^x} = \frac{\lambda}{x+1}$$

8. If X is a poisson variate such that $3P(X=4) = \frac{1}{2} P(X=2) + P(X=0)$

1) Find mean of X 2) $P(X \leq 2)$.

$$3 P(X=4) = \frac{1}{2} P(X=2) + P(X=0)$$

$$\Rightarrow 3 \frac{e^{-\lambda} \cdot \lambda^4}{4!} = \frac{1}{2} \cdot \frac{e^{-\lambda} \cdot \lambda^2}{2!} + \frac{e^{-\lambda} \cdot \lambda^0}{0!}$$

$$\Rightarrow \frac{3\lambda^4}{4!} = \frac{\lambda^2}{4} + 1 \Rightarrow \frac{\lambda^4}{8} = \frac{\lambda^2}{4} + 1$$

$$\Rightarrow \lambda^4 - 2\lambda^2 - 8 = 0 \qquad \text{Let } u = \lambda^2$$

$$\Rightarrow u^2 - 2u - 8 = 0 \Rightarrow u = 4, -2 \Rightarrow \lambda^2 = 4 \text{ or } \lambda^2 = -2 \text{ (-ve Root)}$$

$$\lambda^2 = 4 \Rightarrow \lambda = \pm 2 \Rightarrow \lambda = 2. \qquad (\lambda > 0)$$

Mean $= \lambda = 2$.

2) $P(X \leq 2) = P(X=0) + P(X=1) + P(X=2) = \underline{0.676}$

9. It is know from the past experience that in certain factory 5% of products are defective a sample of 100 items is selected at random Find the probability that exactly 2 products are defective using
i) Binomial ii) Poisson

i) Binomial

$P = 5\% = 0.05 \quad n = 100 \quad f(x) = P(X=x) = {}^nC_x \, p^x q^{n-x}$

$= {}^{100}C_x (0.05)^x (0.95)^{100-x}$

$x = 0, 1, 2, \ldots 100$

$P(X=2) = {}^{100}C_2 (0.05)^2 (0.95)^{98}$

$= \underline{0.0811}$.

ii) Poisson.

$np = \lambda = 0.05 \times 100 = 5$

$f(x) = P(X=x) = \dfrac{e^{-5} \cdot 5^x}{x!} \quad x = 0, 1, 2 \ldots$

$P(X=2) = \dfrac{e^{-5} \cdot 5^2}{2!} = \underline{0.084}$

BIVARIATE RANDOM VARIABLE

Let S be the sample space associated with a random experiment. Let X(t) and Y(t) be two random function, each assign a real number to each outcome $t \in S$, then (X, Y) is called a bivariate random variable. or two dimensional random variable or a bivariate random vector

2-DIMENSIONAL DISCRETE RANDOM VARIABLE

Let (X, Y) be a 2-dimensional random vector, then (X, Y) assumes a finite or a countably infinite number of values then (X, Y) is said to be 2-dimensional discrete random variable.

JOINT PROBABILITY MASS FUNCTION OF (X, Y)

If (X, Y) is a two-dimensional discrete random variable then $P(X=x, Y=y)$ $f(x,y) = P(x,y)$ is called the joint probability mass function of (X, Y) provided

i) $P(x,y) \geq 0$ ii) $\sum\limits_x \sum\limits_y P(x,y) = 1$

Then the triplet $(x, y, P(x,y))$ is called the joint probability distribution of (X, Y).

MARGINAL PROBABILITY MASS FUNCTION

Let X & Y be discrete random variables without joint p.m.f $P(x,y)$ then the marginal probability mass function of X & Y are given by

$$P(x) = P(X=x) = \sum_y P(x,y) \rightarrow \text{Marginal pmf of } X \cdot [P_X(x)]$$

$$P(y) = P(Y=y) = \sum_x P(x,y) \longrightarrow \text{Marginal pmf of } y \; [P_Y(y)]$$

INDEPENDENCE OF X & Y

Two random variables X & Y (discrete) are said to be independent if

$$P(x,y) = P_X(x) \cdot P_Y(y) \text{ or } P(x) \cdot P(y)$$

JOINT CUMULATIVE DISTRIBUTION FUNCTION

$$F(X,Y) = P(X \le x, Y \le y)$$

1. U.Q The joint distribution of a two-dimensional random variable (X,Y) is given by $P(x,y) = c(2x+3y)$ $x = 0,1,2$ $y = 1,2,3$

Find i) the value of c

ii) the marginal distribution iii) Are X and Y independent.

Joint probability mass function of (X,Y)

X \ Y	1	2	3	Total
0	3C	6C	9C	18C
1	5C	8C	11C	24C
2	7C	10C	13C	30C

i) $\sum_x \sum_y P(x,y) = 1$

$18C + 24C + 30C = 1$

$72C = 1 \Rightarrow C = 1/72$

X \ Y	1	2	3	Total $P_X(x)$
0	3/72	6/72	9/72	18/72
1	5/72	8/72	11/72	24/72
2	7/72	10/72	13/72	30/72
Total $P_Y(y)$	15/72	24/72	33/72	1

ii) Marginal pmf of X

X	0	1	2
$P_X(x)$	18/72	24/72	30/72

Marginal pmf of Y

Y	1	2	3
$P_Y(y)$	15/72	24/72	33/72

iii) If X & Y are independent $\Rightarrow P(x,y) = P_X(x) \cdot P_Y(y)$

$P(0,1) = \frac{3}{72}$ $\quad P_X(0) = \frac{18}{72}$ $\quad P_Y(1) = \frac{15}{72}$

$P(0,1) \neq P_X(0) \cdot P_Y(1)$

$\therefore$ X & Y are not independent

2 The joint probability mass function of random variable is given by $P(x,y) = k(x+2y)$ $x = 0,1,2,$ & $y = 0,1,2,3$ Find the following
i) k ii) $P(X\le 1)$ iii) $P(Y \le 2)$ iv) $P(X \le 1, Y \le 2)$ v) $P(X\le 1 / Y\le 2)$
vi) $P(X+Y \le 3)$ vii) marginal distribution viii) Independence.

X \ Y	0	1	2	3	Total
0	0k	2k	4k	6k	12k
1	k	3k	5k	7k	16k
2	2k	4k	6k	8k	20k.

i) $\sum_x \sum_y P(x,y) = 1$

$\Rightarrow 12k + 16k + 20k = 1$

$\Rightarrow 48k = 1$

$\Rightarrow k = 1/48$

Joint p.m.f of (X,Y)

X \ Y	0	1	2	3	Total $P_X(x)$
0	0	$\frac{2}{48}$	$\frac{4}{48}$	$\frac{6}{48}$	$\frac{12}{48}$
1	$\frac{1}{48}$	$\frac{3}{48}$	$\frac{5}{48}$	$\frac{7}{48}$	$\frac{16}{48}$
2	$\frac{2}{48}$	$\frac{4}{48}$	$\frac{6}{48}$	$\frac{8}{48}$	$\frac{20}{48}$
Total $P_Y(y)$	$\frac{3}{48}$	$\frac{9}{48}$	$\frac{15}{48}$	$\frac{21}{48}$	1

ii) $P(X\le 1) = P(X=0) + P(X=1)$

$= P_X(0) + P_X(1) = \frac{12}{48} + \frac{16}{48} = \frac{28}{48}$

iii) $P(Y\le 2) = P(Y=0) + P(Y=1) + P(Y=2)$

$= P_Y(0) + P_Y(1) + P_Y(2)$

$= \frac{3}{48} + \frac{9}{48} + \frac{15}{48} = \frac{27}{48}$

iv) $P(X\le 1, Y\le 2) = P(0,1) + P(0,2) + P(1,1) + P(1,2) + P(0,0) + P(1,0)$

$= \frac{2}{48} + \frac{4}{48} + \frac{3}{48} + \frac{5}{48} + 0 + \frac{1}{48} = \frac{15}{48}$

v) $P(X\le 1/Y\le 2) = \dfrac{P(X\le 1) \cap P(Y\le 2)}{P(Y\le 2)} = \dfrac{15/48}{27/48} = \frac{15}{27}$

vi) $P(X+Y\le 3) = P(0,1) + P(1,0) + P(0,2) + P(2,0) + P(0,3) + P(3,0)$
$+ P(1,1) + P(1,2) + P(2,1)$

$= \frac{2}{48} + \frac{1}{48} + \frac{4}{48} + \frac{6}{48} + \frac{3}{48} + \frac{5}{48} + \frac{2}{48} + \frac{4}{48} = \frac{27}{48}$

Marginal p.m.f of X

X	0	1	2
$P_X(x)$	$12/48$	$16/48$	$20/48$

Marginal p.m.f of Y

Y	0	1	2	3
$P_Y(y)$	$3/48$	$9/48$	$15/48$	$21/48$

Independence

$P(0,1) = \frac{2}{48}$

$P_X(0) = \frac{12}{48}$ $\quad P_Y(1) = \frac{9}{48}$

$P(0,1) \neq P_X(0)\, P_Y(1)$

U.Q

3. Two balls are selected at random from a box containing 3 red, 2 green and 4 white balls. If X and Y are the no: of red balls and green balls respectively included among the two balls drawn from the box. Find i) joint probability ii) marginal probability.

$X \rightarrow$ No: of red balls. $\quad X = 0,1,2,3$

$Y \rightarrow$ No: of green balls $\quad Y = 0,1,2$

X \ Y	0	1	2	$P_X(x)$
0	$\frac{1}{6}$	$\frac{2}{9}$	$\frac{1}{36}$	$\frac{15}{36}$
1	$\frac{1}{3}$	$\frac{1}{12}$	0	$\frac{18}{36}$
2	$\frac{1}{12}$	0	0	$\frac{3}{36}$
3	0	0	0	0
$P_Y(y)$	$\frac{21}{36}$	$\frac{14}{36}$	$\frac{1}{36}$	1

$P(0,0) = \frac{4C_2}{9C_2} = \frac{1}{6}$ $\quad P(0,1) = \frac{2C_1\, 4C_1}{9C_2} = \frac{2}{9}$

$P(0,2) = \frac{2C_2}{9C_2} = \frac{1}{36}$ $\quad P(1,0) = \frac{3C_1\, 4C_1}{9C_2} = \frac{1}{3}$

$P(1,1) = \frac{3C_1\, 2C_1}{9C_2}$ $\quad P(1,2) = 0$

$P(2,0) = \frac{3C_2}{9C_2} = \frac{1}{12}$

Marginal p.m.f of X

X	0	1	2	3
$P_X(x)$	$\frac{15}{36}$	$\frac{18}{36}$	$\frac{3}{36}$	0

Marginal pmf of Y

Y	0	1	2
$P_Y(y)$	$\frac{21}{36}$	$\frac{14}{36}$	$\frac{1}{36}$

EXPECTED VALUES

$$E[X] = \sum_x x P_X(x)$$

$$E[Y] = \sum_y y P_Y(y)$$

$$E[XY] = \sum_x \sum_y xy \, P(x,y)$$

1. Let X and Y be two random variables each taking three values −1, 0 & 1 and having the joint probability distribution. Find $E[X]$, $E[Y]$ & $E[XY]$

Y \ X	−1	0	1	$P_Y(y)$
−1	0	0·1	0·1	0·2
0	0·2	0·2	0·2	0·6
1	0	0·1	0·1	0·2
$P_X(x)$	0·2	0·4	0·4	1

$$E[X] = \sum_x x P_X(x)$$

$$= -1 \times 0{\cdot}2 + 0 \times 0{\cdot}4 + 1 \times 0{\cdot}4 = \underline{\underline{0{\cdot}2}}$$

$$E[Y] = \sum_y y P_Y(y)$$

$$= -1 \times 0{\cdot}2 + 0 \times 0{\cdot}6 + 1 \times 0{\cdot}2 = \underline{\underline{0}}$$

$$E[XY] = \sum_x \sum_y xy \, P(x,y)$$

$$E[XY] = -1 \times -1 \times 0 + -1 \times 0 \times 0{\cdot}1 + -1 \times 1 \times 0{\cdot}1 + 0 \times -1 \times 0{\cdot}2 + 0 \times 0 \times 0{\cdot}2 .$$
$$0 \times 1 \times 0{\cdot}2 + 1 \times -1 \times 0 + 1 \times 0 \times 0{\cdot}1 + 1 \times 1 \times 0{\cdot}1 = 0 .$$